

On Two -Way Nested Classification Random Model

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Abstract

Nested design is a specific kind of experimental design in which each level of each element occurs with all levels of different factors without relationships. The aim of this article is to estimate the variance components of the interferometry model by two methods and to compare them and obtain the least measurement error for these estimators, and depending on the viscosity chemical experiment we obtained the least measurement line through the comparison between the two methods. we conclude that, the variance components estimated by the maximum penalized likelihood method are less than the variance components estimated by the maximum likelihood method. Conversely, the mean squared error of the parameters estimated by the maximum likelihood method is less than the mean squared error of the parameters estimated by the maximum penalized likelihood method.

Thus, the parameters estimated using maximum penalized likelihood method have the least measurement error, and therefore are more accurate and acceptable than the parameters estimated using the maximum likelihood method.

Keywords: Two-Way Nested Design, Analysis of Variance Method (ANOVA), Maximum Likelihood Method (ML), Maximum Penalized Likelihood Method (MPL), Random Effects.

Introduction

Nested model is a type of experimental model in which each component level happens with all levels of the other factors without interactions. For example, in certain examinations, the levels of one component B will not be same across all levels of factor A; instead, each level of factor A will include various levels of factor B; hence, levels of B are considered to be nested inside levels of factor A, [9].

Nevertheless, the fundamental premise for estimating variance components has been and continues to be [2], comparing quadratic formulas of observations to their anticipated values. The functions in the ANOVA- Table are obvious candidates for such functions. Whose was interested in aggregating weights from the woolen industry's carding process, and [17] who studied captures of different kinds in repeated hauls of plankton nets.

The nested model has been dissected considerably for example: Daniel discussed the estimation of components of variance for nested models, [2]. Ganguli, studied properties and notes on nested sampling, [3]. Klotz, at al. defined the mean square efficiency of estimators of variance Components, [5]. Bainbridge estimated the variance components of staggered nested designs, [1]. Pignatiello used examples of two stage nested models, [11]. Harville estimated the variance components by Maximum likelihood approaches and to related problems, [4]. Rao and Kleffe estimated the variance components with applications, [12]. Lamar and Zirk used application industry in Nested model, [7]. Liu and Batson measured the profit in the manufacture of steel tubes based on the overlapping model, [8]. Song and Robert estimated the community mean for the model overlapping with chemical and physical applications, [15]. Rasch and Masata estimated variance components by using many approach [12]. Kori compared between two method of variance components [6].

The purpose of this article is to estimate and compare the variance components of the nested model using two approaches. we conclude that, the variance components estimated by the maximum penalized likelihood method are less than the variance components estimated by the maximum likelihood method. Conversely, the mean squared error of the parameters estimated by the maximum likelihood method is less than the mean squared error of the parameters estimated by the maximum penalized likelihood method.

Thus, the parameters estimated using maximum penalized likelihood method have the least measurement error, and therefore are more accurate and acceptable than the parameters estimated using the maximum likelihood method.

Mathematical Model

The mathematical random model of two-way nested model can be written as follows:

$$y_{ijk} = \mu + \omega_i + \psi_{j(i)} + \varepsilon_{ijk}, \quad i = 1, \dots, r; j = 1, \dots, s; k = 1, \dots, t, \quad (1)$$

where y_{ijk} is the response variable for k th observation equivalent to the j th level of factor Ω within the i th level of factor Ψ , μ is the constant mean, ω_i is the random effect of the i th factor level of Ω , $\psi_{j(i)}$ is the random effect of the i th factor level of Ψ within the i th factor level of Ω and ε_{ijk} is the term of random error. Assume that the parameters have the independently distributions as follows $\omega_i \sim N(0, \sigma_\omega^2)$, $\psi_{j(i)} \sim N(0, \sigma_\psi^2)$, $\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$ and $-\infty < \mu < \infty$,

for all $\sigma_\omega^2, \sigma_\psi^2$ and σ_ε^2 are variance components of the two-way nested model.

The analysis of the variance table of the two-way nested model, which can be written as follow:

Table 1: The ANOVA Table of The Two-Way Nested Model

Source of Variance	D.F.	SS	MS	E(MS)
Random ω	$r - 1$	SS_ω	MS_ω	$st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2$
Random ψ	$r(s - 1)$	SS_ψ	MS_ψ	$t\sigma_\psi^2 + \sigma_\varepsilon^2$
Error ε	$rs(t - 1)$	SS_ε	MS_ε	σ_ε^2
SS_{total}	$rst - 1$			

where

$$\begin{aligned} SS_\omega &= rt \sum_{j=1}^s (\bar{y}_{.j} - \bar{y}_{...})^2, \\ SS_\psi &= r \sum_{i=1}^r \sum_{j=1}^s (\bar{y}_{ij.} - \bar{y}_{.j.})^2, \\ SS_\varepsilon &= \sum_{i=1}^r \sum_{j=1}^s \sum_{k=1}^t (\bar{y}_{.j.} + \bar{y}_{i..} - \bar{y}_{ij.} - \bar{y}_{i.k})^2, \\ \bar{y}_{...} &= \frac{1}{rst} \sum_{i=1}^r \sum_{j=1}^s \sum_{k=1}^t y_{ijk}, \bar{y}_{i..} = \frac{1}{rs} \sum_{j=1}^r \sum_{k=1}^s y_{ijk}, \end{aligned}$$

$$\bar{y}_{.j.} = \frac{1}{rt} \sum_{i=1}^r \sum_{k=1}^t y_{ijk}, \bar{y}_{ij.} = \frac{1}{t} \sum_{k=1}^t y_{ijk} \text{ and } \bar{\Psi}_{i.k} = \frac{1}{s} \sum_{j=1}^s y_{ijk}.$$

The sufficient statistics of $\bar{y}_{...}$, SS_{ω} , SS_{ψ} and SS_{ε} have the following distributions:

$$\begin{aligned} \bar{y}_{...} &\sim N\left(\mu, \frac{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2}{rst}\right), \\ SS_{\omega} &\sim (st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) \chi^2(\alpha), \\ SS_{\psi} &\sim (t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) \chi^2(\beta) \text{ and } \\ SS_{\varepsilon} &\sim (\sigma_{\varepsilon}^2) \chi^2(\gamma). \end{aligned}$$

for all $\alpha = r - 1$, $\beta = r(s - 1)$ and $\gamma = rs(t - 1)$ are the degree of freedom for random model (1).

Estimation Parameters of Two-Way Nested Random Model

Maximum Likelihood Method

The likelihood function method for the random model (1) as follow:

$$L(y) = \frac{\exp\left\{-\frac{1}{2}\left(\frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^2} + \frac{rst(\bar{y}_{...} - \mu)^2}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)}\right)\right\}}{(2\pi)^{\frac{1}{2}rst}(\sigma_{\varepsilon}^2)^{\frac{1}{2}\gamma}(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^{\frac{1}{2}\beta}(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^{\frac{1}{2}\alpha}} \quad (2)$$

for all $\alpha = r - 1$, $\beta = r(s - 1)$ and $\gamma = rs(t - 1)$ are the degree of freedom for random model (1).

We can rewrite the result (2) by using the Ln-likelihood function as follows:

$$\begin{aligned} \ln L(\Psi) = & -\frac{1}{2}\left[rst \ln(2\pi) + \gamma \ln(\sigma_{\varepsilon}^2) + \beta \ln(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) + \alpha \ln(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) + \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^2} + \right. \\ & \left. \frac{rst(\bar{y}_{...} - \mu)^2}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)}\right] \quad (3) \end{aligned}$$

Now, we derivative the result (3) for parameter μ and equal the derivative to zero, we have that

$$\begin{aligned} \frac{\partial \ln L(\Psi)}{\partial \mu} &= -\frac{1}{2} [2rst(\bar{y}_{...} - \mu)(-1)] = 0, \text{ we have that } \\ \hat{\mu} &= \bar{y}_{...} \quad (4) \end{aligned}$$

We can rewrite equation (3) as follows:

$$\ln L(\Psi) = -\frac{1}{2}\left[rst \ln(2\pi) + \gamma \ln(\sigma_{\varepsilon}^2) + \beta \ln(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) + \alpha \ln(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) + \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)} + \frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^2}\right] \quad (5)$$

Now, we derivative the result (3) for parameters σ_{ε}^2 , σ_{ψ}^2 and σ_{ω}^2 respectively and equal the derivative to zero, we have that

$$\frac{\partial \ln L(\Psi)}{\partial \sigma_{\varepsilon}^2} = \frac{\gamma}{\sigma_{\varepsilon}^2} + \frac{\beta}{t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} + \frac{\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} - \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} - \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} - \frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^4} = 0$$

which can write as

$$\begin{aligned} \frac{\gamma}{\sigma_{\varepsilon}^2} + \frac{\beta}{t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} + \frac{\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} &= \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} + \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} + \frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^4} \quad (6) \\ \frac{\partial \ln L(\Psi)}{\partial \sigma_{\psi}^2} &= \frac{t\beta}{t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} + \frac{t\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} - \frac{tSS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} - \frac{tSS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} = 0 \end{aligned}$$

which can write as

$$\begin{aligned} \frac{\beta}{t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} + \frac{\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} &= \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} + \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} \quad (7) \\ \frac{\partial \ln L(\Psi)}{\partial \sigma_{\omega}^2} &= \frac{st\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} - \frac{stSS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} = 0 \end{aligned}$$

which can write as

$$\frac{\alpha}{st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} = \frac{SS_{\omega}}{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} \quad (8)$$

substitute equation (8) in (7), we have that

$$\frac{\beta}{t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2} = \frac{SS_{\psi}}{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2} \quad (9)$$

substitute equation (8) and (9) in (6), we have that

$$\frac{\gamma}{\sigma_\varepsilon^2} = \frac{SS_\varepsilon}{\sigma_\varepsilon^4} \rightarrow \hat{\sigma}_\varepsilon^2 = \frac{SS_\varepsilon}{\gamma} \quad (10)$$

from equation (9), we have that

$$\beta = \frac{SS_\psi}{t\sigma_\psi^2 + \sigma_\varepsilon^2} \rightarrow \hat{\sigma}_\psi^2 = \frac{1}{t} \left(\frac{SS_\psi}{\beta} - \frac{SS_\varepsilon}{\gamma} \right) \quad (11)$$

substitute equations (10) and (11) in (8), we have that

$$\alpha = \frac{SS_\omega}{st\sigma_\omega^2 + t \left[\frac{1}{t} \left(\frac{SS_\psi}{\beta} - \frac{SS_\varepsilon}{\gamma} \right) \right] + \frac{SS_\varepsilon}{\gamma}} \rightarrow \hat{\sigma}_\omega^2 = \frac{1}{st} \left(\frac{SS_\omega}{\alpha} - \frac{SS_\psi}{\beta} \right) \quad (12)$$

Theorem 1: The variance components estimated by the maximum likelihood method of model (1) are unbiased.

Proof:

$$\begin{aligned} E(\hat{\sigma}_\varepsilon^2) - \sigma_\varepsilon^2 &= E\left(\frac{SS_\varepsilon}{\gamma}\right) - \sigma_\varepsilon^2 = \frac{1}{\gamma} E(SS_\varepsilon) - \sigma_\varepsilon^2 \\ &= \frac{1}{\gamma} \gamma \sigma_\varepsilon^2 - \sigma_\varepsilon^2 = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} E(\hat{\sigma}_\psi^2) - \sigma_\psi^2 &= E\left[\frac{1}{t} \left(\frac{SS_\psi}{\beta} - \frac{SS_\varepsilon}{\gamma} \right)\right] - \sigma_\psi^2 = \frac{1}{t} [(t\sigma_\psi^2 + \sigma_\varepsilon^2) - \sigma_\varepsilon^2] - \sigma_\psi^2 \\ &= \frac{1}{t} [t\sigma_\psi^2] - \sigma_\psi^2 = \sigma_\psi^2 - \sigma_\psi^2 = 0 \end{aligned} \quad (14)$$

$$\begin{aligned} E(\hat{\sigma}_\omega^2) - \sigma_\omega^2 &= E\left[\frac{1}{st} \left(\frac{SS_\omega}{\alpha} - \frac{SS_\psi}{\beta} \right)\right] - \sigma_\omega^2 \\ &= \frac{1}{st} [st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2 - t\sigma_\psi^2 - \sigma_\varepsilon^2] - \sigma_\omega^2 = 0 \\ &= \frac{1}{st} [st\sigma_\omega^2] - \sigma_\omega^2 = 0 \end{aligned} \quad (15)$$

where $\alpha = r - 1$, $\beta = r(s - 1)$ and $\gamma = rs(t - 1)$.

Theorem 2: The variance for variance components estimated by the maximum-likelihood method is biased.

$$var(\hat{\sigma}_\varepsilon^2) = var\left(\frac{SS_\varepsilon}{\gamma}\right) = \frac{var(SS_\varepsilon)}{\gamma^2} = \frac{2\gamma\sigma_\varepsilon^4}{\gamma^2} = \frac{2\sigma_\varepsilon^4}{\gamma} \quad (16)$$

$$\begin{aligned} var(\hat{\sigma}_\psi^2) &= var\left(\frac{1}{t} \left(\frac{SS_\psi}{\beta} - \frac{SS_\varepsilon}{\gamma} \right)\right) = \frac{1}{t^2} \left(\frac{var(SS_\psi)}{\beta^2} + \frac{var(SS_\varepsilon)}{\gamma^2} \right) \\ &= \frac{2\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{t^2\beta^2} + \frac{2\gamma\sigma_\varepsilon^4}{t^2\gamma^2} = \frac{2}{t^2} \left(\frac{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\beta} + \frac{\sigma_\varepsilon^4}{\gamma} \right) \end{aligned} \quad (17)$$

$$\begin{aligned} var(\hat{\sigma}_\omega^2) &= var\left(\frac{1}{st} \left(\frac{SS_\omega}{\alpha} - \frac{SS_\psi}{\beta} \right)\right) = \frac{1}{(st)^2} \left(\frac{var(SS_\omega)}{\alpha^2} + \frac{var(SS_\psi)}{\beta^2} \right) \\ &= \frac{2\alpha(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{s^2t^2\alpha^2} + \frac{2\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{s^2t^2\beta^2} \\ &= \frac{2}{s^2t^2} \left(\frac{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\alpha} + \frac{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\beta} \right) \end{aligned} \quad (18)$$

Maximum Penalized Likelihood Method"

The maximum penalized likelihood function is defined as follows:

$$L = y + p(\sigma_\omega^2, \sigma_\psi^2, \sigma_\varepsilon^2) \quad (19)$$

The Ln-penalized likelihood function can be written as follows:

$$\ln(L) = L(y) + \ln p(\sigma_\omega^2, \sigma_\psi^2, \sigma_\varepsilon^2) \quad (20)$$

Where

$L(y)$: Ln-penalized likelihood function,

$\ln p(\sigma_\omega^2, \sigma_\psi^2, \sigma_\varepsilon^2)$: the penalty term".

The extra penalty is calculated using the Ln-penalized likelihood function for the Bayesian prior density. The prior distribution of $\mu, \sigma_\omega^2, \sigma_\psi^2, \sigma_\varepsilon^2$ depend on reference Tiao and Tan can be defined as follows: " [16]

$$p(\sigma_\omega^2, \sigma_\psi^2, \sigma_\varepsilon^2) \propto \frac{1}{(s\sigma_\omega^2 + \sigma_\psi^2 + \sigma_\varepsilon^2)(t\sigma_\psi^2 + \sigma_\varepsilon^2)\sigma_\varepsilon^2} \quad (21)$$

$$\ln L(\Psi) = -\frac{1}{2} \left[rst \ln(2\pi) + \gamma \ln(\sigma_\varepsilon^2) + \beta \ln(t\sigma_\psi^2 + \sigma_\varepsilon^2) + \alpha \ln(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2) + \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)} + \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)} + \frac{SS_\varepsilon}{\sigma_\varepsilon^2} + \frac{rst(\bar{y}_{...} - \mu)^2}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)} \right] - \ln(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2) - \ln(t\sigma_\psi^2 + \sigma_\varepsilon^2) - \ln(\sigma_\varepsilon^2) \quad (23)$$

Now, we will maximize the equation (23) with respect to μ and equal to zero, we have that

$$\frac{\partial \ln L(\Psi)}{\partial \mu} = -\frac{1}{2} [2rst(\bar{y}_{...} - \mu)(-1)] = 0, \text{ we have that} \\ \hat{\mu} = \bar{y}_{...} \quad (24)$$

We can rewrite equation (23) as follows:

$$\ln L(\Psi) = -\frac{1}{2} \left[rst \ln(2\pi) + \gamma \ln(\sigma_\varepsilon^2) + \beta \ln(t\sigma_\psi^2 + \sigma_\varepsilon^2) + \alpha \ln(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2) + \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)} + \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)} + \frac{SS_\varepsilon}{\sigma_\varepsilon^2} \right] - \ln(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2) - \ln(t\sigma_\psi^2 + \sigma_\varepsilon^2) - \ln(\sigma_\varepsilon^2) \quad (25)$$

Now, we will maximize the equation (25) with respect to σ_ε^2 , σ_ψ^2 and σ_ω^2 respectively and equal to zero, we have that

$$\frac{\partial \ln L(\Psi)}{\partial \sigma_\varepsilon^2} = \frac{\gamma+2}{\sigma_\varepsilon^2} + \frac{\beta+1}{t\sigma_\psi^2 + \sigma_\varepsilon^2} + \frac{(\alpha+1)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} - \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} - \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} - \frac{SS_\varepsilon}{\sigma_\varepsilon^4} = 0$$

which can written as

$$\frac{\gamma+2}{\sigma_\varepsilon^2} + \frac{\beta+2}{t\sigma_\psi^2 + \sigma_\varepsilon^2} + \frac{(\alpha+2)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} = \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} + \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} + \frac{SS_\varepsilon}{\sigma_\varepsilon^4} \quad (26) \\ \frac{\partial \ln L(\Psi)}{\partial \sigma_\psi^2} = \frac{t(\beta+2)}{t\sigma_\psi^2 + \sigma_\varepsilon^2} + \frac{t(\alpha+2)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} - \frac{tSS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} - \frac{tSS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} = 0$$

which can written as

$$\frac{\beta+2}{t\sigma_\psi^2 + \sigma_\varepsilon^2} + \frac{(\alpha+2)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} = \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} + \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} \quad (27) \\ \frac{\partial \ln L(\Psi)}{\partial \sigma_\omega^2} = \frac{st(\alpha+2)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} - \frac{stSS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} = 0$$

which can written as

$$\frac{(\alpha+2)}{st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2} = \frac{SS_\omega}{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} \quad (28)$$

substitute equation (8) in (27), we have that

$$\frac{\beta+2}{t\sigma_\psi^2 + \sigma_\varepsilon^2} = \frac{SS_\psi}{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2} \quad (29)$$

substitute equation (28) and (29) in (26), we have that

$$\frac{\gamma+2}{\sigma_\varepsilon^2} = \frac{SS_\varepsilon}{\sigma_\varepsilon^4} \rightarrow \hat{\sigma}_\varepsilon^2 = \frac{SS_\varepsilon}{\gamma+2} \quad (30)$$

from equation (29), we have that

$$\beta+2 = \frac{SS_\psi}{t\sigma_\psi^2 + \sigma_\varepsilon^2} \rightarrow \hat{\sigma}_\psi^2 = \frac{1}{t} \left(\frac{SS_\psi}{\beta+2} - \frac{SS_\varepsilon}{\gamma+2} \right) \quad (31)$$

substitute equations (30) and (31) in (28), we have that

$$(\alpha+2) = \frac{SS_\omega}{st\sigma_\omega^2 + t \left[\frac{1}{t} \left(\frac{SS_\psi}{\beta+2} - \frac{SS_\varepsilon}{\gamma+2} \right) \right] + \frac{SS_\varepsilon}{\gamma+2}} \rightarrow \hat{\sigma}_\omega^2 = \frac{1}{st} \left(\frac{SS_\omega}{\alpha+2} - \frac{SS_\psi}{\beta+2} \right) \quad (32)$$

Theorem 3: The variance components estimated by the penalized maximum likelihood method of model (1) are biased.

Proof:

$$E(\hat{\sigma}_\varepsilon^2) - \sigma_\varepsilon^2 = E\left(\frac{SS_\varepsilon}{\gamma+2}\right) - \sigma_\varepsilon^2 = \frac{1}{\gamma+2} E(SS_\varepsilon) - \sigma_\varepsilon^2 \\ = \frac{1}{\gamma+2} (\gamma\sigma_\varepsilon^2) - \hat{\sigma}_\varepsilon^2 \quad (33) \\ E(\hat{\sigma}_\psi^2) - \sigma_\psi^2 = E\left[\frac{1}{t} \left(\frac{SS_\psi}{\beta+2} - \frac{SS_\varepsilon}{\gamma+2} \right)\right] - \sigma_\psi^2$$

$$\begin{aligned}
 &= \frac{1}{t(\beta+2)} E(SS_\psi) - \frac{1}{t(\gamma+2)} E(SS_\varepsilon) - \sigma_\psi^2 \\
 &= \frac{1}{t(\beta+2)} \beta(t\sigma_\psi^2 + \sigma_\varepsilon^2) - \frac{1}{t(\gamma+2)} (\gamma\sigma_\varepsilon^2) - \sigma_\psi^2 \\
 &= \frac{t(\beta\gamma+\beta)\sigma_\psi^2 + 2(\beta-\gamma)\sigma_\varepsilon^2}{t(\beta+2)(\gamma+2)} - \sigma_\psi^2 \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 E(\hat{\sigma}_\omega^2) - \sigma_\omega^2 &= \frac{1}{st} E\left(\frac{SS_\omega}{\alpha+2} - \frac{SS_\psi}{\beta+2}\right) - \sigma_\omega^2 = \frac{1}{st} \left(\frac{E(SS_\omega)}{\alpha+2} - \frac{E(SS_\psi)}{\beta+2}\right) - \sigma_\omega^2 \\
 &= \frac{1}{st} \left(\frac{\alpha(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)}{\alpha+2} - \frac{\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)}{\beta+2}\right) - \sigma_\omega^2 \\
 &= \frac{\alpha(\beta+2)(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2) - \beta(\alpha+2)(t\sigma_\psi^2 + \sigma_\varepsilon^2)}{st(\alpha+2)(\beta+2)} - \sigma_\omega^2 \tag{35}
 \end{aligned}$$

Theorem 4: The variance for variance components estimated by the penalized maximum-likelihood method is biased.

Proof:

$$var(\hat{\sigma}_\varepsilon^2) = var\left(\frac{SS_\varepsilon}{\gamma+2}\right) = \frac{var(SS_\varepsilon)}{(\gamma+2)^2} = \frac{2\gamma^2\sigma_\varepsilon^4}{(\gamma+2)^2} = \frac{2\gamma^2\sigma_\varepsilon^4}{(\gamma+2)^2} \tag{36}$$

$$\begin{aligned}
 var(\hat{\sigma}_\psi^2) &= var\left(\frac{1}{t} \left(\frac{SS_\psi}{\beta+2} - \frac{SS_\varepsilon}{\gamma+2}\right)\right) = \frac{1}{t^2} \left(\frac{var(SS_\psi)}{(\beta+2)^2} + \frac{var(SS_\varepsilon)}{(\gamma+2)^2}\right) \\
 &= \frac{2\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{t^2(\beta+2)^2} + \frac{2\gamma\sigma_\varepsilon^4}{t^2(\gamma+2)^2} = \frac{2}{t^2} \left(\frac{\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{(\beta+2)^2} + \frac{\gamma\sigma_\varepsilon^4}{(\gamma+2)^2}\right) \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 var(\hat{\sigma}_\omega^2) &= var\left(\frac{1}{st} \left(\frac{SS_\omega}{\alpha+2} - \frac{SS_\psi}{\beta+2}\right)\right) = \frac{1}{(st)^2} \left(\frac{var(SS_\omega)}{(\alpha+2)^2} + \frac{var(SS_\psi)}{(\beta+2)^2}\right) \\
 &= \frac{2}{s^2t^2} \left(\frac{2\alpha(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{(\alpha+2)^2} + \frac{2\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{(\beta+2)^2}\right) \tag{38}
 \end{aligned}$$

The theorem that follows is used to demonstrate the link among variance component estimators and to examine the two methods via mean square error.

Theorem 5: The mean squared error of the variance components estimated by the Maximum likelihood method is less than the mean squared error of the variance components estimated by the maximum likelihood of penalty method.

$$a- MSE(\hat{\sigma}_{\varepsilon,ML}^2) \leq MSE(\hat{\sigma}_{\varepsilon,MPL}^2), \tag{39}$$

$$b- MSE(\hat{\sigma}_{\psi,ML}^2) \leq MSE(\hat{\sigma}_{\psi,MPL}^2) \text{ and} \tag{40}$$

$$c- MSE(\hat{\sigma}_{\omega,ML}^2) \leq MSE(\hat{\sigma}_{\omega,MPL}^2). \tag{41}$$

Proof:

$$\begin{aligned}
 a- MSE(\hat{\sigma}_{\varepsilon,ML}^2) &= var(\hat{\sigma}_{\varepsilon,ML}^2) = \frac{2\sigma_\varepsilon^4}{\gamma} \\
 MSE(\hat{\sigma}_{\varepsilon,MPL}^2) &= var(\hat{\sigma}_{\varepsilon,MPL}^2) + [E(\hat{\sigma}_{\varepsilon,MPL}^2) - \sigma_{\varepsilon,MPL}^2]^2 \\
 &= \frac{2\gamma\sigma_\varepsilon^4}{(\gamma+2)^2} + \left[\frac{1}{\gamma+2} (\gamma\sigma_\varepsilon^2) - \sigma_\varepsilon^2\right]^2 \\
 \therefore MSE(\hat{\sigma}_{\varepsilon,ML}^2) &\leq MSE(\hat{\sigma}_{\varepsilon,MPL}^2) \\
 b- MSE(\hat{\sigma}_{\psi,ML}^2) &= var(\hat{\sigma}_{\psi,ML}^2) = \frac{2}{t^2} \left(\frac{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\beta} + \frac{\sigma_\varepsilon^4}{\gamma}\right) \\
 MSE(\hat{\sigma}_{\psi,MPL}^2) &= var(\hat{\sigma}_{\psi,MPL}^2) + [E(\hat{\sigma}_{\psi,MPL}^2) - \sigma_{\psi,MPL}^2]^2 \\
 &= \frac{2}{t^2} \left(\frac{\beta(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{(\beta+2)^2} + \frac{\gamma\sigma_\varepsilon^4}{(\gamma+2)^2}\right) + \left[\frac{t(\beta\gamma+\beta)\sigma_\psi^2 + 2(\beta-\gamma)\sigma_\varepsilon^2}{t(\beta+2)(\gamma+2)} - \sigma_\psi^2\right]^2 \\
 \therefore MSE(\hat{\sigma}_{\psi,ML}^2) &\leq MSE(\hat{\sigma}_{\psi,MPL}^2) \\
 c- MSE(\hat{\sigma}_{\omega,ML}^2) &= var(\hat{\sigma}_{\omega,ML}^2) = \frac{2}{s^2t^2} \left(\frac{(st\sigma_\omega^2 + t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\alpha} + \frac{(t\sigma_\psi^2 + \sigma_\varepsilon^2)^2}{\beta}\right)
 \end{aligned}$$

$$MSE(\hat{\sigma}_{\omega, MPL}^2) = var(\hat{\sigma}_{\omega, MPL}^2) + [E(\hat{\sigma}_{\omega, MPL}^2) - \sigma_{\omega, MPL}^2]^2$$

$$= \frac{2}{s^2 t^2} \left(\frac{\alpha(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2}{(\alpha+2)^2} + \frac{\beta(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2}{(\beta+2)^2} \right) + \left[\frac{\alpha(\beta+2)(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2) - \beta(\alpha+2)(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)}{st(\alpha+2)(\beta+2)} - \sigma_{\omega}^2 \right]^2$$

$$\therefore MSE(\hat{\sigma}_{\omega, ML}^2) \leq MSE(\hat{\sigma}_{\omega, MPL}^2)$$

The Practical Side

A chemical experiment was used in a laboratory to measure viscosity, where the experiment took place in several stages: A batch of 500 kg (100 g) was sampled, after which the sample was prepared (dilution with 20 g of material and a solvent) and finally the viscosity was measured. Therefore, our goal of the practical application is to analyze and compare the total measurement error and develop some algorithms necessary to achieve the reduction of the total measurement error depending on the two methods.[10]

Table 2: Results of Repeatability and Reproducibility Study

Sample	S ₁			S ₂			S ₃			S ₄		
Dilution	D ₁	D ₂	D ₃	D ₁	D ₂	D ₃	D ₁	D ₂	D ₃	D ₁	D ₂	D ₃
Test 1	62.1	62.9	64	61.4	65.7	64.7	68.3	70.8	62.5	63.5	60.2	64.2
Test 2	67.9	60.2	57.1	66.4	51.5	62.7	58.3	69.4	65	60.4	55.9	54.5
Test 3	65.7	61.8	58.5	64.1	71.7	62.3	71.2	71.9	63.1	60.1	63.4	61.1

The mathematical model for the data shown above can be written as follows:

$$y_{ijk} = \mu + \omega_i + \psi_{j(i)} + \varepsilon_{ijk}, \quad i = 1, \dots, 4; j = 1, \dots, 3; k = 1, \dots, 3,$$

where i = 4 and j = 3 represent the number of levels for the sample and dilution, respectively. k = 3 is the number of repeated measurements. Table 3 shows the outcomes of data treatment via the ANOVA technique depend on the parameters which estimated by maximum likelihood method.

Table 3: The ANOVA Table of The Two-Way Nested Model by Using ML

Source of Variance	D.F.	SS	MS	E(MS)
Sample ω	3	204.45	68.15	68.13
Dilution ψ	8	143.39	17.92	17.91
Error ε	24	300.17	12.51	12.51
SS_{total}	35	648.01		

$$\hat{\sigma}_{\varepsilon, ML}^2 = \frac{SS_{\varepsilon}}{\gamma} = 12.51$$

$$\hat{\sigma}_{\psi, ML}^2 = \frac{1}{t} \left(\frac{SS_{\psi}}{\beta} - \frac{SS_{\varepsilon}}{\gamma} \right) = 1.8$$

$$\hat{\sigma}_{\omega, ML}^2 = \frac{1}{st} \left(\frac{SS_{\omega}}{\alpha} - \frac{SS_{\psi}}{\beta} \right) = 5.58$$

$$MSE(\hat{\sigma}_{\varepsilon, ML}^2) = \frac{2\sigma_{\varepsilon}^4}{\gamma} = 13.04$$

$$MSE(\hat{\sigma}_{\psi, ML}^2) = \frac{2}{t^2} \left(\frac{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2}{\beta} + \frac{\sigma_{\varepsilon}^4}{\gamma} \right) = 10.36$$

$$MSE(\hat{\sigma}_{\omega, ML}^2) = \frac{2}{s^2 t^2} \left(\frac{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2}{\alpha} + \frac{(t\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)^2}{\beta} \right) = 14.39$$

Table 3 shows the outcomes of data treatment via the ANOVA technique depend on the parameters which estimated by maximum likelihood method.

Table 4: The ANOVA Table of The Two-Way Nested Model by Using MPL

Source of Variance	D.F.	SS	MS	E(MS)
Sample ω	3	204.45	68.15	34.044
Dilution ψ	8	143.39	17.92	14.244
Error ε	24	300.17	12.51	11.545
SS_{total}	35	648.01		

$$\hat{\sigma}_{\varepsilon, MPL}^2 = \frac{SS_{\varepsilon}}{\gamma+2} = 11.545$$

$$\hat{\sigma}_{\psi, MPL}^2 = \frac{1}{t} \left(\frac{SS_{\psi}}{\beta+2} - \frac{SS_{\epsilon}}{\gamma+2} \right) = 0.93$$

$$\hat{\sigma}_{\omega, MPL}^2 = \frac{1}{st} \left(\frac{SS_{\omega}}{\alpha+2} - \frac{SS_{\psi}}{\beta+2} \right) = 2.2$$

$$MSE(\hat{\sigma}_{\epsilon, MPL}^2) = \frac{2\gamma\sigma_{\epsilon}^4}{(\gamma+2)^2} + \left[\frac{1}{\gamma+2} (\gamma\sigma_{\epsilon}^2) - \sigma_{\epsilon}^2 \right]^2 = 14.25$$

$$MSE(\hat{\sigma}_{\psi, MPL}^2) = \frac{2}{t^2} \left(\frac{\beta(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\beta+2)^2} + \frac{\gamma\sigma_{\epsilon}^4}{(\gamma+2)^2} \right) + \left[\frac{t(\beta\gamma + \beta)\sigma_{\psi}^2 + 2(\beta-\gamma)\sigma_{\epsilon}^2}{t(\beta+2)(\gamma+2)} - \sigma_{\psi}^2 \right]^2 = 31.364$$

$$MSE(\hat{\sigma}_{\omega, MPL}^2) = \frac{2}{s^2 t^2} \left(\frac{2\alpha(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\alpha+2)^2} + \frac{2\beta(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\beta+2)^2} \right) + \left[\frac{\alpha(\beta+2)(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\epsilon}^2) - \beta(\alpha+2)(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)}{st(\alpha+2)(\beta+2)} - \sigma_{\omega}^2 \right]^2 = 33.787$$

we conclude through this study, the variance components estimated by the maximum penalized likelihood method are less than the variance components estimated by the maximum likelihood method. Conversely, the mean squared error of the parameters estimated by the maximum likelihood method is less than the mean squared error of the parameters estimated by the maximum penalized likelihood method.

Thus, the parameters estimated using maximum penalized likelihood method have the least measurement error, and therefore are more accurate and acceptable than the parameters estimated using the maximum likelihood method.

Conclusion

In this article, we conclude as follows:

- 1- The variance components estimated by the maximum likelihood method are as follows:

- a- $\hat{\sigma}_{\epsilon, ML}^2 = \frac{SS_{\epsilon}}{\gamma} = 12.51$

- b- $\hat{\sigma}_{\psi, ML}^2 = \frac{1}{t} \left(\frac{SS_{\psi}}{\beta} - \frac{SS_{\epsilon}}{\gamma} \right) = 1.8$

- c- $\hat{\sigma}_{\omega, ML}^2 = \frac{1}{st} \left(\frac{SS_{\omega}}{\alpha} - \frac{SS_{\psi}}{\beta} \right) = 5.58$

- 2- The variance components estimated by the maximum likelihood method are unbiased.

- 3- The variance for variance components estimated by the maximum-likelihood method is biased.

- 4- The mean square error for estimated parameters by the maximum penalized likelihood method as follows.

- a- $MSE(\hat{\sigma}_{\epsilon, ML}^2) = \frac{2\sigma_{\epsilon}^4}{\gamma} = 13.04$

- b- $SE(\hat{\sigma}_{\psi, ML}^2) = \frac{2}{t^2} \left(\frac{(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{\beta} + \frac{\sigma_{\epsilon}^4}{\gamma} \right) = 10.36$

- c- $MSE(\hat{\sigma}_{\omega, ML}^2) = \frac{2}{s^2 t^2} \left(\frac{(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{\alpha} + \frac{(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{\beta} \right) = 14.39$

- 5- The variance components estimated by the maximum penalized likelihood method are as follows:

- a- $\hat{\sigma}_{\epsilon, MPL}^2 = \frac{SS_{\epsilon}}{\gamma+2} = 11.545$

- b- $\hat{\sigma}_{\psi, MPL}^2 = \frac{1}{t} \left(\frac{SS_{\psi}}{\beta+2} - \frac{SS_{\epsilon}}{\gamma+2} \right) = 0.93$

- c- $\hat{\sigma}_{\omega, MPL}^2 = \frac{1}{st} \left(\frac{SS_{\omega}}{\alpha+2} - \frac{SS_{\psi}}{\beta+2} \right) = 2.88$

- 6- The variance components estimated by the maximum penalized likelihood method are biased.

- 7- The variance for variance components estimated by the maximum penalized likelihood method is biased.

- 8- The mean square error for estimated parameters by the maximum penalized likelihood method as follows.

- a- $MSE(\hat{\sigma}_{\epsilon, MPL}^2) = \frac{2\gamma\sigma_{\epsilon}^4}{(\gamma+2)^2} + \left[\frac{1}{\gamma+2} (\gamma\sigma_{\epsilon}^2) - \sigma_{\epsilon}^2 \right]^2 = 14.25$

- b- $MSE(\hat{\sigma}_{\psi, MPL}^2) = \frac{2}{t^2} \left(\frac{\beta(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\beta+2)^2} + \frac{\gamma\sigma_{\epsilon}^4}{(\gamma+2)^2} \right) + \left[\frac{t(\beta\gamma + \beta)\sigma_{\psi}^2 + 2(\beta-\gamma)\sigma_{\epsilon}^2}{t(\beta+2)(\gamma+2)} - \sigma_{\psi}^2 \right]^2 = 31.364$

- c- $MSE(\hat{\sigma}_{\omega, MPL}^2) = \frac{2}{s^2 t^2} \left(\frac{2\alpha(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\alpha+2)^2} + \frac{2\beta(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)^2}{(\beta+2)^2} \right) + \left[\frac{\alpha(\beta+2)(st\sigma_{\omega}^2 + t\sigma_{\psi}^2 + \sigma_{\epsilon}^2) - \beta(\alpha+2)(t\sigma_{\psi}^2 + \sigma_{\epsilon}^2)}{st(\alpha+2)(\beta+2)} - \sigma_{\omega}^2 \right]^2 = 33.787$

- 9- The parameters estimated using maximum penalized likelihood method have the least measurement error, and therefore are more accurate and acceptable than the parameters estimated using the maximum likelihood method.

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