

Product of Total and Range Fuzzy Edge Labeling of Interval-Valued Fuzzy Graph

D. Geetha ^a, S. Santhosh Kumar ^{b*}, N. Karthika ^c, V. Mathan Kumar ^d,
Jessalet Ann Mathew ^e

^a Research Scholar, Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Bharathiar University, Coimbatore, Tamil Nadu, India.

E-mail: dgeetha@rmv.ac.in, Orcid: <https://orcid.org/0009-0009-2886-1316>

^{b*} Assistant Professor, Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Bharathiar University, Coimbatore, Tamil Nadu, India.

E-mail: santhoshkumars@rmv.ac.in, Orcid: <https://orcid.org/0000-0003-2276-3706>

^c Assistant Professor, Department of Mathematics, Kangeyam Institute of Commerce, Kangeyam, Bharathiar University, Tamil Nadu, India. E-mail: nk.maths@kic.edu.in,

Orcid: <https://orcid.org/0009-0006-8880-9016>

^d Professor, Department of Mathematics, Laxminarayana Arts & Science College for Women, Dharmapuri, Periyar University, Tamil Nadu, India. E-mail: laxminarayanawomencollege@lncwarts.com,

Orcid: <https://orcid.org/0009-0004-7076-6412>

^e Research Scholar, Department of Mathematics, Sri Vasavi College, Erode, Bharathiar University, Tamil Nadu, India. E-mail: annjes95@gmail.com, Orcid: <https://orcid.org/0009-0009-6721-770X>

Abstract

The fuzzy graph theory can be described as a generalization of the classical graph theory wherein the structures of the graphs are defined in uncertain form. Interval-valued fuzzy graphs are generalizations of fuzzy graphs wherein the membership value of each structure is given in terms of intervals instead of crisp values. However, the existing approaches for labeling edges in fuzzy graphs are mainly either additive or rule-based, and thus they cannot efficiently accommodate complex networks with high uncertainties. To address the challenge, the authors present a new method known as Product of Total and Range Fuzzy Edge Labeling (PTARFEL) for interval-valued fuzzy graphs. The PTARFEL approach applies multiplicative procedure to calculate the membership values of the edges according to the total and range of the membership values of the connected vertices. Interval valued ε neighborhood construction is employed to provide appropriate labeling and to keep the intervals of vertices and edges distinct under various graph settings. The theoretical properties are developed using rigorous proof concerning the fuzzy graph, the interval-valued labeled graph, and the conditions that guarantee consistency of the product-based labeling. The effectiveness of the proposed labeling approach is evaluated for the well-known graph classes including path, star, and cycle graphs. The simulation results confirm the ability of the proposed label to satisfy the fuzzy conditions, different label allocation, and consistency even with perturbation of intervals; hence, improving the performance of the conventional labeling approach. In conclusion, the proposed model is a generalized and flexible framework to represent uncertainty in network structures. Therefore, it generalizes the concept of interval-valued fuzzy graph labeling. This model enables its applicability in decision making, communication networks, and optimization problems where the information is always imprecise.

Keywords: Fuzzy Graph, Fuzzy Path, Fuzzy Cycle, Fuzzy Star, Product of Total and Range Fuzzy Edge Labeling, Interval-Valued Fuzzy Labeled Graph.

Introduction

In many real-world systems, uncertainty is an inherent characteristic, especially when information is incomplete, imprecise, vague, and dynamically changing. While classical graph theory can be used to describe the relationships between objects, it does not have the power to describe the strength of their vertices and their connecting edges if it cannot be represented as an exact number. It is also imperative keep in mind that there is not only this constraint. The fuzzy graph theory extends the graph theory by assigning a membership value to each vertex and edge. Membership value helps in making the graph capable of expressing the uncertainties and vague relationships more effectively. However, in some instances, it is not sufficient to assign just one membership value, as the information may be in the form of intervals rather than a single value. To cope with this problem, Zadeh proposed interval valued fuzzy sets. In these types of graphs, vertices and edges have lower and upper bounds for their membership and are applicable to systems that are not exactly described. They are useful in many other application areas like communication networks, decision-making systems, optimization problems, medical diagnosis, remote sensing, social networks, and other complex environments where the relationship cannot be precisely defined. Interval-valued fuzzy graphs are more suitable for modeling real-world problems as they can model a wider range of uncertainty than ordinary fuzzy graphs.

Graph labeling is yet another significant topic in the field of graph theory and fuzzy graph theory. Graph labeling in the field of fuzzy graph involves the use of certain criteria for assigning the degree of membership to vertices and edges. It helps maintain the uniqueness of the vertices and edges and satisfies the criteria of fuzzy logic. Apart from that, there are already existing labeling concepts such as magic labeling and bi-magic labeling. But most of these methods are primarily based on additive relationships or fixed labeling constraints. They are not always appropriate for interval-valued fuzzy environments, particularly when the edge memberships need to capture the combined effect and the variation between the memberships of their incident vertices. Thus, it would be helpful to have a more flexible labeling technique that can deal with interval uncertainty and maintain the structure of the graph.

In this research, a novel definition termed as the “Product of Total and Range Fuzzy Edge Labeling for Interval-valued Fuzzy Graphs” is introduced. The process here uses multiplication of total and range of fuzzy membership values of connected vertices for determining the fuzzy membership values of the edges. This is important as it considers the contribution of each vertex membership as well as the difference between them. This makes the proposed approach a useful way of determining edge memberships in ambiguous graph structures. This research also adopts ϵ -neighborhood intervals to transform the fuzzy graphs into an interval-valued fuzzy labeled graph, thereby retaining the valid membership values of the fuzzy graph while meeting the unique labeling conditions.

This paper has introduced a novel interval-valued product-based fuzzy edge labeling framework, which is a key contribution. The researchers formally introduce the Product of Total and Range Fuzzy Edge Labeling and further generalize this idea for interval-valued fuzzy graphs. They further prove that some fuzzy graphs can be transformed into interval-valued products of total and range fuzzy edge labeling graphs. Moreover, the paper provides significant theoretical results for standard graph classes, including path graphs, star graphs, and odd cycle graphs. These results indicate that the proposed labeling method maintains fuzzy conditions, allows for different interval values of memberships, and makes fuzzy graph labeling more flexible.

The paper has five structured sections, Section 1 (Introduction) explains the limitations of existing fuzzy labelling methods and introduces the idea of a more flexible approach of multiplicative edge labelling; Section 2 (Review of Literature) provides background information on other advancements made in graph theory and the specific gaps in finding multiplicative type edge labelling; Section 3 (Definitions of Terminologies Used in the Study) presents the basic mathematical foundation of the work including definition of interval valued fuzzy graph and the ϵ -neighborhood construction used to convert it; Section 4 (Interval Valued Product of Total and Range Edge Fuzzy Labeling Graph) is the core section that explains the proposed PTARFEL framework, its properties and the validation of the proposed framework over various standard graph classes such as paths, stars and odd cycles; Section 5 (Conclusion) concludes the paper by summarizing the contributions made by the proposed framework in modelling uncertainty and providing directions for future research.

Review of Literature

Another very important branch of fuzzy graph theory involves the treatment of uncertainties by giving membership values to the nodes and edges of a graph. The works done by Akram (2012) on the application of membership values based on intervals in interval-valued fuzzy line graphs have been able to extend the structure of ordinary fuzzy graph models and initiate interval valued fuzzy concepts in graph models for the first time (Akram, 2012). Fuzzy graphs

have also been applied to computational and imaging problems by Ciesielski et al. (2012), who applied fuzzy connectedness in graph-cut image segmentation (Ciesielski et al., 2012).

In 2016, Pramanik et al. worked on interval-valued fuzzy planar graphs and gave a study of the structural properties of these graphs (Pramanik et al., 2016). They concluded that interval-valued fuzzy graphs are more effective than normal fuzzy graphs in the modeling of uncertain planar relations. In 2016, Akram and Adeel introduced m-polar fuzzy labeling graphs and their applications to extend the concept of fuzzy labeling to a more general fuzzy situation (Akram & Adeel, 2016). This contribution by Akram and Adeel, (2016) extended the concept of fuzzy labeling through m-polar fuzzy labeling graphs.

Kishore Kumar et al., in 2017, have investigated magic labeling on interval-valued intuitionistic fuzzy graphs (Kishore Kumar et al., 2017). Their work has significance since they linked the notion of labeling theory with interval-valued intuitionistic fuzzy graph structure. In 2017, Rashmanlou and Borzooei developed some new notions in relation to fuzzy labeling graphs and made some theoretical contribution in this field (Rashmanlou & Borzooei, 2017; Borzooei et al., 2016). Devi, in 2018, analyzed some graphs from the point of view of fuzzy anti-magic labeling, thereby developing fuzzy labeling techniques for other types of graphs (Devi, 2018).

Sahoo and Pal, in 2018, have worked on edge-irregular intuitionistic fuzzy graphs and elaborated upon the notion of irregularity in the context of fuzzy graph models (Sahoo & Pal, 2018). In 2018, Saravanan et al. have analyzed the notion of integrity of fuzzy graphs in order to measure the vulnerability and strength of the graph (Saravanan et al., 2018). Samanta and Sarkar, in 2018, have discussed the notion of generalized fuzzy graphs that widened the scope of fuzzy graph theory in terms of representation of graph in a more flexible way (Samanta & Sarkar, 2018). The present study considers the product-based graph operations in the environment of interval-valued fuzzy, same as considered by Rashmanlou et al. in 2018 where they studied the product of interval-valued fuzzy graphs and degree (Rashmanlou et al., 2018).

The structural properties of various m-polar interval-valued fuzzy graphs have been investigated by Bera & Pal, (2020). The study revealed that interval-valued fuzzy graph theory continues to evolve and grow complex and multi-dimensional environments can be considered. Afsharmanesh and Borzooei, (2022) investigated domination in fuzzy incidence graphs where there exist edges (Afsharmanesh & Borzooei, 2022). The results indicated that the use of edge-based fuzzy relations is essential in analyzing domination in graphs. Qiang et al., (2022) found out that interval-valued fuzzy graphs can play an important role in energy sector management, thus demonstrating that such models have applications in decision-making and management systems (Qiang et al., 2022).

Raut and Pal, (2022) developed fuzzy intersection graphs through a geometrical approach (Raut & Pal, 2022). The authors indicated that geometrical relations could provide a better modeling capability of fuzzy graph structures. Xu et al., (2022) examined the applications of interval-valued intuitionistic fuzzy graphs and provided a framework for the practical applications of such models under uncertainty conditions (Xu et al., 2022). Zhang et al., (2023) stressed the modeling capabilities of fuzzy graphs in computational intelligence through fuzzy representation learning on graphs (Zhang et al., 2023).

Adhikari et al., (2023) have mentioned the operations on interval-valued picture fuzzy graphs and their applications (Adhikari et al., 2023). They have also extended the theory of interval-valued fuzzy graphs to picture fuzzy environment. Arif et al., (2023) have investigated the decision-making with multiple attributes in the case of interval-valued picture fuzzy graphs and argued that the use of the advanced fuzzy graph model helps in dealing with the complex decision-making problems with uncertain information (Arif et al., 2023).

Based on the review of the literature, there has been significant attention to fuzzy graphs, interval-valued fuzzy graphs, intuitionistic fuzzy graphs, picture fuzzy graphs, domination, integrity, operations on graphs, and labeling schemes recently. But most of the current labeling schemes have focused on additive, magic, anti-magic, or irregular conditions. Multiplicative edge labeling schemes have focused on the sum of the membership values of vertices of interval-valued fuzzy graphs but not on the range of the membership value of vertices, where range indicates the set of vertices whose membership values belong to an interval. In this regard, this paper presents the Product of Total and Range Fuzzy Edge Labeling scheme for the interval-valued fuzzy graphs and proves its feasibility for some well-known graphs such as paths, stars, and odd cycles.

Definitions of Terminologies Used in the Study

Definition 3.1 A fuzzy graph $G = (V, \rho, \delta)$ is defined by a pair of functions (ρ, δ) , where $\rho: V \rightarrow [0,1]$ is a fuzzy subset of a non-empty set V , and $\delta: V \times V \rightarrow [0,1]$ is a symmetric fuzzy relation on ρ , such that in equation (1).

$$\delta(uv) \leq \rho(u) \wedge \rho(v) \quad (1)$$

for all $u, v \in V$ and $(u, v) \in E \subseteq V \times V$.

Definition 3.2. Let $G = (V, \rho, \delta)$ be a fuzzy graph, a fuzzy path P_n in G is a sequence of distinct vertices $v_0, v_1, \dots, v_n$ such that $\delta(v_{i-1}, v_i) > 0, 1 \leq i \leq n$.

Definition 3.3. A fuzzy path p_n in which $v_0 = v_n$ and $n \geq 3$ is called a fuzzy cycle C_n of length n .

Definition 3.4. (Borzoeei et al., 2016) A fuzzy Star graph consists of two vertex sets V and U with $|V| = 1$ and $|U| = \theta$ such that $\rho(v, u_i) > 0$ and $\rho(u_i, u_{i+1}) = 0$ for $1 \leq i \leq \theta$.

Definition 3.5. (Adhikari et al., 2023) An interval-valued fuzzy graph of a graph G is a pair $G^* = (A, B)$, where $A = [\rho_v^-, \rho_v^+]$ is an interval-valued fuzzy set on V and $B = [\delta_e^-, \delta_e^+]$ is an interval valued? fuzzy relation on E such that $\delta_e^-(uv) \leq \min\{\rho_A^-(u), \rho_A^+(v)\}$, $\delta_e^+(uv) \leq \min\{\rho_A^+(u), \rho_A^-(v)\}$, for all $uv \in E$.

Definition 3.6. (Ciesielski et al., 2012) A graph $\tilde{G} = (\rho_v, \delta_e)$ is said to be a fuzzy labeling graph if $\rho(v): V \rightarrow [0, 1]$ and $\delta(uv): V \times V \rightarrow [0, 1]$ are bijective mappings such that the membership values of nodes and edges are distancing equation (2).

$$\delta_e(uv) \leq \min\{\rho_v(u) \wedge \rho_v(v)\}, \forall uv \in V \quad (2)$$

Definition 3.7. (Adhikari et al., 2023) A graph $\tilde{G} = (A, B)$ is said to be an Interval-Valued Fuzzy Labeling Graph if the functions $\rho_v^-, \rho_v^+: A \rightarrow [0, 1]$ and $\delta_e^-, \delta_e^+: B \rightarrow [0, 1]$ are assigned such that all lower and upper membership values of vertices and edges are distinct. Here, ρ_v^- and ρ_v^+ denote the lower and upper limits of the interval-valued membership function of the vertices, respectively. Similarly, δ_e^- and δ_e^+ denote the lower and upper limits of the interval-valued membership function of edges respectively.

Definition 3.8. (Ciesielski et al., 2012) An interval $[\rho - \varepsilon, \rho + \varepsilon]$ is said to be an ε -neighborhood of any membership value ρ (corresponding to any node or edge) for any ε satisfying the following conditions:

1. $\varepsilon \leq \min_{i,j} \{\rho_v(v_i), \rho_e(e_{ij})\}$,
2. $\varepsilon \leq 1 - \max_{i,j} \{\rho_v(v_i), \rho_e(e_{ij})\}$,
3. $\varepsilon \neq d(\rho(u), \rho(v))$

$$(\text{or}) \quad \frac{1}{2} d(\rho(u), \rho(v))$$

Where $d(\rho(u), \rho(v)) = |\rho(u) - \rho(v)|$ and $\rho(u), \rho(v)$ denote the membership values of nodes or edges.

Theorem 3.1. Any fuzzy graph can be converted into an interval-valued fuzzy labeling graph.

Proof: As it is aware, not all fuzzy graphs are necessarily fuzzy labeling graphs. In order to make the labeling of fuzzy graphs, it gives interval-valued membership to all the vertices and edges of the graph. In order to do this, it considers the ε -neighborhood of all the vertices and edges. It claims that in this way an interval-valued fuzzy labeling graph can be formed.

There are Only Three Cases Possible for any Fuzzy Graph

Case (i): If all the vertices and edges have the same membership value, and if the total number of vertices and edges is n , then select n different values of ε as defined before.

In this way, by making use of different ε -neighborhoods, obtain mutually disjoint intervals. Hence, the graph obtained is an interval-valued fuzzy labeling graph.

Case (ii): In case only a few vertices and edges have identical membership values, initially enumerate the membership values alongside the respective vertices and edges. From this list, select a set of identical values and give the identical value, $\varepsilon = \varepsilon_1$, to this set. Remove the members of this set from the list. Select another set of identical membership values from the remaining list and give the value, $\varepsilon = \varepsilon_2$, to this set. Continue with this process until all the elements are removed from the list. In this way, ε -neighborhood intervals will be generated for each ε_i , where $i = 1, 2, \dots$. Thus, assigning these intervals to the respective vertices and edges will generate the interval-valued fuzzy graph satisfying interval-valued fuzzy labeling criteria.

Case (iii): However, if the membership values of the nodes and edges are all different, then only one value of ε is required. Since the ε -neighborhoods of all the nodes and edges would be different, a fuzzy graph could be converted to an interval-valued fuzzy labeling graph under any conditions.

Example: Suppose that there is a fuzzy labeling graph G consisting of four nodes and edges with membership values assigned to each of them.

Constructing the corresponding interval-valued fuzzy labeling graph G^* , if all the nodes and edges have unique membership values, then taking only one value of ϵ , obtain different ϵ -neighborhoods of nodes and edges.

Hence, in any case, the fuzzy graph can be converted to an interval-valued fuzzy labeling graph.

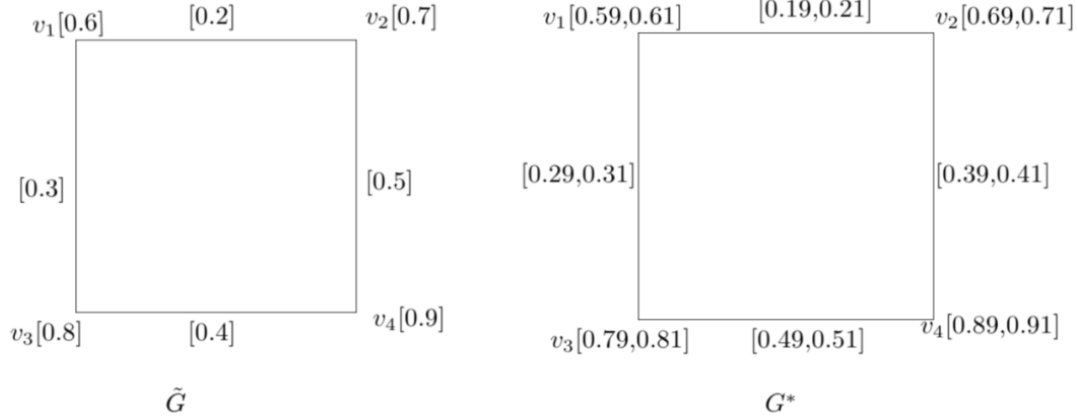


Fig. 1: Fuzzy Labeling Graph $\tilde{G}$ into an Interval-Valued Fuzzy Labeling Graph

A graph $\tilde{G}$ is formed from all vertices and edges, with distinct membership in the set $W = \{0.6, 0.7, 0.8, 0.9, 0.2, 0.3, 0.4, 0.5\}$, which corresponds to $v_1, v_2, v_3, v_4, v_1v_2, v_1v_3, v_3v_4, v_4v_2$. set $\epsilon = 0.01$ for W and strike off that element from the list $\tilde{G}$ which are in W . The interval-valued fuzzy labeling graph is shown in fig. 1 as G^* .

Table 1: Notations and Symbols Used in the Proposed PTARFEL Framework

Symbol	Description
$G = (V, \rho, \delta)$	A fuzzy graph
V	Non-empty set of vertices
ρ	Membership function of vertices
δ	Membership function of edges
$u, v \in V$	Vertices of the graph
P_n	Fuzzy path
C_n	Fuzzy cycle of length n
$\tilde{G} = (V, \rho, \delta)$	Interval-valued fuzzy graph
$\rho(u) = [\rho^-(u), \rho^+(u)]$	Interval membership of vertex u
$\delta(uv) = [\delta^-(uv), \delta^+(uv)]$	Interval membership of edge (u, v)
$[\rho^-(u), \rho^+(u)]$	Lower and upper bounds of vertex membership
$[\delta^-(uv), \delta^+(uv)]$	Lower and upper bounds of edge membership
ϵ	Small positive number (epsilon)

The key symbols and notations adopted in the study of Product of Total and Range Fuzzy Edge Labeling (PTARFEL) in interval – valued fuzzy graph are presented in table 1. Throughout the paper, the following symbols are used to represent the basic graph structures, membership functions, interval representations, and labeling parameters used in the theoretical formulations, proofs, and examples.

Interval Valued Product of total and Range Edge Fuzzy Libeling Graph

Definition 4.1. A fuzzy graph $G = (V, E)$ admits a product of total and range fuzzy edge labeling in equation 3 if there exist bijections $\rho(v): V \rightarrow [0, I]$ and $\delta(uv): V \times V \rightarrow [0, I]$ satisfying $\delta(uv) < \min\{\rho(u), \rho(v)\}$ and

$$\delta(uv) = |\rho(u)^2 - \rho(v)^2| \quad (3)$$

for all $uv \in E$.

Definition 4.2. Let $G^* = (V, E)$ be an interval-valued fuzzy labeled graph. Then G^* is said to satisfy the product of total and range interval-valued fuzzy edge labeling if for every edge $(uv) \in E$, the following lower and upper conditions hold in equations 4 - 7:

$$\delta^-(uv) < \min\{\rho^-(u), \rho^-(v)\} \quad (4)$$

and

$$\delta^-(uv) = |\rho^-(u)^2 - \rho^-(v)^2| \quad (5)$$

$$\delta^+(uv) < \min\{\rho^+(u), \rho^+(v)\} \quad (6)$$

and

$$\delta^+(uv) = |\rho^+(u)^2 - \rho^+(v)^2| \quad (7)$$

Example:

Let $G = (V, E)$ be a fuzzy graph with vertex set $V = \{v_1, v_2, v_3\}$ and edge set $E = \{v_1v_2, v_2v_3, v_1v_3\}$.

Define the vertex labeling function

$$\rho: V \rightarrow [0, 1]$$

Define the vertex labels as follows:

$$\rho(v_i) = \left[\frac{i}{100}, \frac{i}{10} \right] \quad i = 1, 2, 3.$$

$$\rho(v_1) = [0.01, 0.1] \quad \rho(v_2) = [0.02, 0.2] \quad \rho(v_3) = [0.03, 0.3]$$

Define the edge labels as follows:

$$\delta(uv): V \times V \rightarrow [0, 1]$$

$$\delta^-(u_i v_i) = |\rho^-(u_i)^2 - \rho^-(v_i)^2|$$

$$\delta^+(u_i v_{i+1}) = |\rho^+(u_i)^2 - \rho^+(v_{i+1})^2|$$

$$[\delta^-(u_1 v_2), \delta^+(u_1 v_2)] = [0.0003, 0.03],$$

$$[\delta^-(u_1 v_3), \delta^+(u_1 v_3)] = [0.0008, 0.08], [\delta^-(u_2 v_3), \delta^+(u_2 v_3)] = [0.0005, 0.05].$$

Check the fuzzy condition

$$\delta^-(uv) < \min\{\rho^-(u), \rho^-(v)\} \quad (8)$$

$$\delta^+(uv) < \min\{\rho^+(u), \rho^+(v)\} \quad (9)$$

Hence, the condition is satisfied in equations 8 and 9

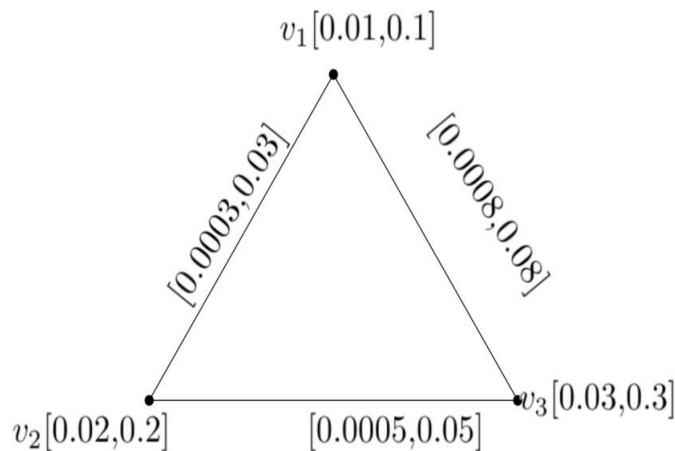


Fig. 2: Interval-valued Fuzzy Edge Labeling Graph

Theorem 4.1. Every fuzzy graph that can have a Product of Total and Range Fuzzy Edge Labeling can be turned into an Interval-Valued Product of Total and Range Fuzzy Edge Labeling graph, as shown in fig. 2.

Proof: In a fuzzy graph that supports Product of Total and Range Fuzzy Edge Labeling, each node has a unique membership value. Let the membership value of nodes be $\rho(v_i)$ and $\rho(u_i)$ for each u_i, v_i in $V(G)$. The membership value of each edge u_iv_i in $E(G)$ is given by equation 10. $\delta(uv) = |\rho(u)^2 - \rho(v)^2|$ (10)

Since the graph admits Product-of-Total and Range labeling, all edge membership values are distinct.

Now, find equation 11

$$m = \min_i \{\rho(v_i)\}, M = \max_i \{\rho(v_i)\} \quad (11)$$

Choose any ε satisfying the additional conditions

$$M + \varepsilon \leq 1 \quad \text{and} \quad m - \varepsilon \geq 0$$

and also satisfying the conditions of Definition 2.4. replace the membership value of each node with $[\rho(u)_i - \varepsilon, \rho(v_i) + \varepsilon]$.

Further, define the interval-valued edge membership for each edge u_iv_i as in equation 12

$$\delta(u_iv_i) = |(\rho(u_i)^2 - \varepsilon) - (\rho(v_i)^2 + \varepsilon)| \quad (12)$$

Since the original fuzzy graph admits distinct membership values for each node and edge, and since the int, choosing start equation script epsilon satisfying the required conditions always produces mutually disjoint intervals. Moreover, because the function order of the edge labels remains unchanged under a small perturbation by ε . Hence, the Product of the total and Range property is preserved in interval form. Thus, the obtained graph becomes an Interval-Valued Product of Total and Range Fuzzy Edge Labeling graph.

Example

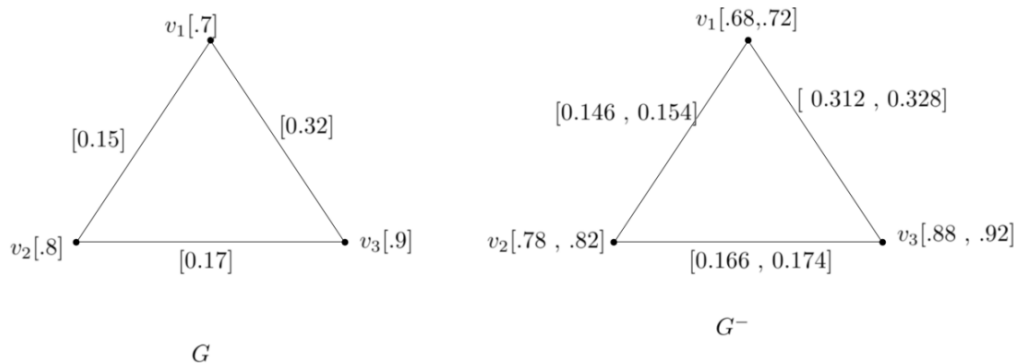


Fig. 3: Fuzzy Labeled Graph G and Interval-Valued Fuzzy Edge-Labeled Graph $\tilde{G}$

In this example, calculate the interval-valued product of the total and range fuzzy labeled graph G using the multiplication of the total and range fuzzy edge graph G. In fig. 3, the value of ε is given as 0.2, and calculate the ε neighborhood intervals for G'. It has been seen that this product of total and range fuzzy graph fulfills all the required conditions for the product of total and range fuzzy edge labeling in an interval-valued fuzzy graph.

Theorem 4.2. Every fuzzy labeling of a graph can always be converted into an Interval-valued Total and Range Product Fuzzy Edge Labeled graph. However, it is possible that not all the resulting interval memberships are pairwise disjoint.

Proof: Let $G = (V, E)$ be a fuzzy labeled graph in which distinct membership values are assigned to vertices and edges.

To construct the interval-valued version, choose a small $\varepsilon \geq 0$, such that the modified values remain in $[0, 1]$. Now replace each vertex membership $\rho(u_i)$ and $\rho(v_i)$ by the interval $[\rho(u_i) - \varepsilon, \rho(v_i) + \varepsilon]$ and define the edge intervals according to the product of the total and the range fuzzy edge labeling formula. Since the original graph satisfies the required product condition, the same relation holds for the corresponding lower and upper bounds of the intervals.

Hence, the obtained graph satisfies the conditions of the product of the total and range interval-valued fuzzy edge labeling. However, because the length of the interval depends on the chosen ε , two intervals may overlap if ε is large. Therefore, the resulting intervals are not necessarily mutually disjoint.

Lemma: Let $G = (V, E)$ be a fuzzy graph with distinct vertex membership values $\rho(v_i)$.

Let $d = \min_{i \neq j} |\rho(u_i) - \rho(v_j)|$.

If construct interval memberships as

$$[\rho(u)_i - \varepsilon, \rho(v_i) + \varepsilon]$$

Then,

- If $\varepsilon < \frac{d}{2}$, all the intervals are mutually disjoint.
- If $\varepsilon \geq \frac{d}{2}$, at least two intervals overlap.

Proof: Since the vertex membership values are distinct, the minimum distance between any two of them is $d > 0$.
Two intervals

$$[\rho(u_i) - \varepsilon, \rho(v_i) + \varepsilon] \text{ and } [\rho(u_j) - \varepsilon, \rho(v_j) + \varepsilon]$$

are disjoint if and only if

$$|\rho(v_i) - \rho(v_j)| \geq 2\varepsilon.$$

Because the smallest possible value of $|\rho(v_i) - \rho(v_j)|$ is d , all intervals remain disjoint precisely. when

$$d > 2\varepsilon.$$

that is,

$$\varepsilon < \frac{d}{2}.$$

If $\varepsilon \geq \frac{d}{2}$, then at least one pair of intervals overlaps.

Theorem 4.3. The P_n can be transformed into an interval-valued fuzzy graph which satisfies the product of total and range interval-valued fuzzy edge labeling conditions.

Proof: Let $P_n = (V, E)$ be a fuzzy path graph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$.

In fig. 3, $\tilde{G}$ is a fuzzy graph, whose vertex and edge membership function is defined in equations 13 and 14

$$\rho(u_i) = \frac{i}{n+1}, 1 \leq i \leq n \quad (13)$$

$$\delta(uv) = |\rho(u_i)^2 - \rho(u_{i+1})^2|, 1 \leq i \leq n-1 \quad (14)$$

Choose $\varepsilon = 0.02$, construct the corresponding ε -neighborhood intervals for G^* as follows in equations 15 - 19:

For each vertex v_i .

$$\rho_{G^*}(u_i) = |\rho_G(u_i) - \varepsilon, \rho_G(u_i) + \varepsilon| \quad (15)$$

and for each edge uv ,

$$\delta_{G^*}^-(u_i v_i) = |\rho_{G^*}^-(u_i)^2 - \rho_{G^*}^-(v_i)^2| \quad (16)$$

$$\delta_{G^*}^+(u_i v_i) = |\rho_{G^*}^+(u_i)^2 - \rho_{G^*}^+(v_i)^2| \quad (17)$$

and satisfied

$$\delta_{G^*}^-(u_i v_i) < \min\{\rho_{G^*}^-(u_i), \rho_{G^*}^-(v_i)\} \quad (18)$$

$$\delta_{G^*}^+(u_i v_i) = \min(\rho_{G^*}^+(u_i), \rho_{G^*}^+(v_i)) \quad (19)$$

Thus, the obtained interval-valued fuzzy graph G^* satisfies all the conditions of the product of total and range fuzzy edge labeling.

Example: Consider the fuzzy path graph P_4 .

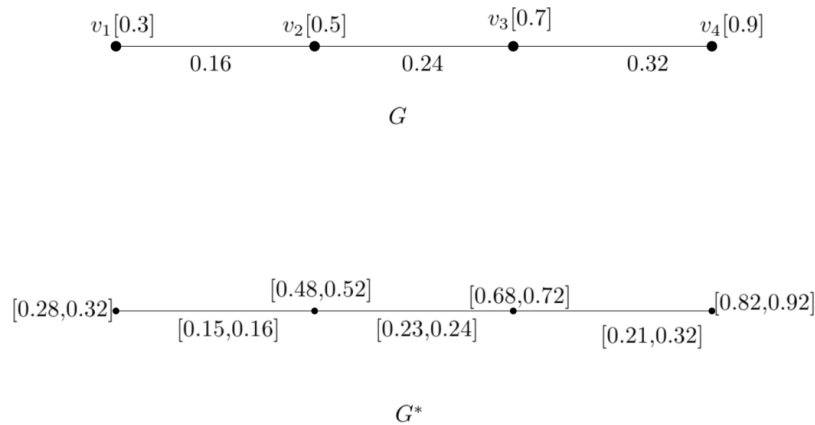


Fig. 4: PTARF Edge Labeled Graph and Corresponding IVPTARF

Theorem 4.4. For any $n \geq 4$, a fuzzy labeled star graph $S_{I,n}$ is always an interval-valued product of the total and range Fuzzy edge labeling graph in fig. 4.

Proof: Let $S_{I,n}$ be the star graph with vertex set

$$V(S_{I,n}) = \{v, u_1, u_2, \dots, u_n\}$$

and edge set

$$E(S_{I,n}) = \{vu_1, vu_2, \dots, vu_n\}.$$

Choose $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$\varepsilon_1 = 0.001, \varepsilon_2 = 0.01, \varepsilon_1 < \varepsilon_2,$$

Where $\varepsilon_1 < \varepsilon_2$ and $n \geq 4$.

Define the lower and upper vertex membership functions as follows:

For $i = 1, 2, 3$

$$\rho^-(u_i) = [2(n+1) - i]\varepsilon_2 - \varepsilon_1,$$

$$\rho^+(u_i) = [2(n+1) - i]\varepsilon_2 + \varepsilon_1.$$

For $4 \leq i \leq n$

$$\rho^-(u_i) = [2(n+1) - i - 1]\varepsilon_2 - \varepsilon_1,$$

$$\rho^+(u_i) = [2(n+1) - i - 1]\varepsilon_2 + \varepsilon_1,$$

$$\rho^-(v) = \frac{\sum \rho^-(u_i)}{n} - (n+1)\delta_1,$$

$$\rho^+(v) = \frac{\sum \rho^+(u_i)}{n} - (n+1)\delta_1.$$

Since $\varepsilon_1 < \varepsilon_2$, it has

$$\rho^-(u_i) < \rho^+(u_i),$$

for all $i = 1, 2, \dots, n$ and hence each vertex is assigned a valid interval-valued membership.

Further, for $n \geq 4$, all such interval memberships are distinct and lie within $[0, 1]$.

Define the membership values for edges,

$$\delta^-(uv_i) = |\rho^-(u)^2 - \rho^-(v_i)^2|$$

$$\delta^+(uv_i) = |\rho^+(u)^2 - \rho^+(v_i)^2|.$$

and satisfied in equations (20) and (21).

$$\delta^-(uv_i) < \min\{\rho^-(u), \rho^-(v_i)\}, \quad (20)$$

$$\delta^+(uv_i) < \min\{\rho^+(u), \rho^+(v_i)\} \quad (21)$$

Hence, S_I admits the product of the total and range Interval-Valued Fuzzy edge labeling for all $n \geq 4$.

Example:

Consider an interval-valued labeled fuzzy star graph $S_{I,3}$ which is the product of total and range fuzzy edge labeling in fig. 5.

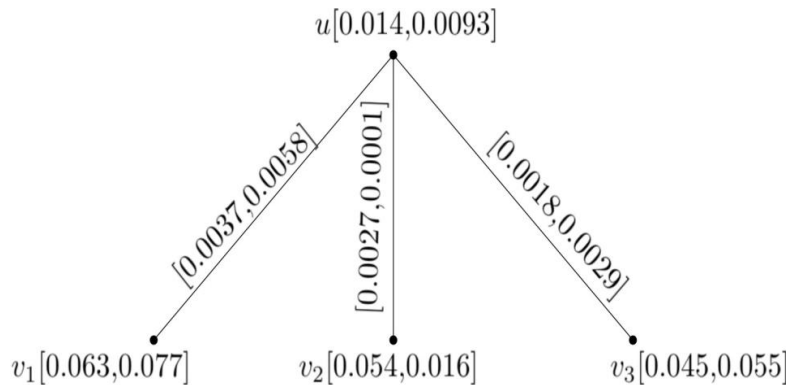


Fig. 5: Interval-valued Labeled Fuzzy Star Graph

Theorem 4.5. Every fuzzy cycle graph C_n with an odd number of vertices admits a product of total and range interval-valued fuzzy edge labeling in fig. 6.

Proof: Let $C_n = (V, E)$ be a fuzzy cycle graph with $V = \{v_1, v_2, \dots, v_n\}$, where n is odd and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.

For $n \geq 3$, take $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.01$.

For $n \geq 4$, take $\varepsilon_1 = 0.002$, $\varepsilon_2 = 0.02$,

Since n is odd, taken $n = 2k + 1$.

Define the vertex membership functions as follows:

For even-indexed vertices,

$$\rho^-(v_{2i}) = (2n + i)\varepsilon_1, \rho^+(v_{2i}) = (2n + i)\varepsilon_2, 1 \leq i \leq k.$$

Let

$$m_2 = \min\{\rho^+(v_{2i}) : 1 \leq i \leq k\}, m_1 = \min\{\rho^-(v_{2i}) : 1 \leq i \leq k\},$$

For odd indexed vertices,

$$\rho^-(v_{2i-1}) = m_1 - i\varepsilon_1, \rho^+(v_{2i-1}) = m_2 - i\varepsilon_2, 1 \leq i \leq k.$$

Clearly,

$$0 \leq \rho^-(v) \leq \rho^+(v) \leq 1, \forall v \in V.$$

Now define the edge membership functions as follows:

$$\delta^-(u_i, u_{i+1}) = |\rho^-(u_i)^2 - \rho^-(u_{i+1})^2|$$

$$\delta^+(u_i, u_{i+1}) = |\rho^+(u_i)^2 - \rho^+(u_{i+1})^2|$$

Since $0 < \rho^-(v_i) < 1$ and $0 < \rho^+(v_i) < 1$,

Then

$$\delta^-(u_i u_{i+1}) < \min\{\rho^-(u_i), \rho^-(u_{i+1})\} \quad (22)$$

$$\delta^+(u_i u_{i+1}) < \min\{\rho^+(u_i), \rho^+(u_{i+1})\} \quad (23)$$

Hence, C_n satisfies the product of the total and range interval-valued fuzzy edge labeling conditions in equations (22) and (23).

Hence, from the above discussion, it is concluded that every odd cycle is an interval-valued fuzzy magic graph. However, this property does not necessarily hold for an even cycle. When the same procedure is applied to an even cycle, certain edges may obtain interval membership values that violate the required conditions of the product of the total and range fuzzy edge labeling.

Example: consider the fuzzy cycle C_7 .

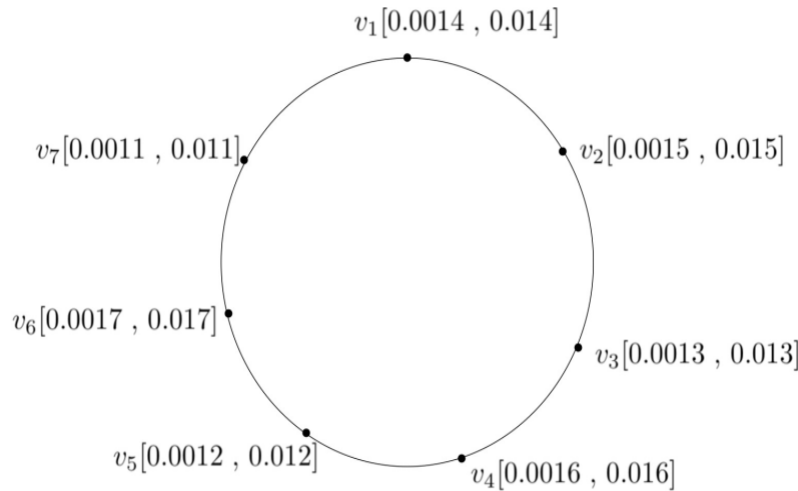


Fig. 6: Interval-valued Labeled Fuzzy Cycle Graph

The edge labels are defined as,

$$\begin{aligned}\delta^-(e_1) &= 0.0001, \delta^+(e_1) = 0.0116, \delta^-(e_2) = 0.00022, \delta^+(e_2) = 0.0224, \\ \delta^-(e_3) &= 0.00034, \delta^+(e_3) = 0.034, \delta^-(e_4) = 0.00044, \delta^+(e_4) = 0.0448, \\ \delta^+(e_4) &= 0.0448, \delta^-(e_5) = 0.00058, \delta^+(e_5) = 0.058, \delta^+(e_6) = 0.0672.\end{aligned}$$

Conclusion

In this paper, an algorithm called Product of Total and Range Fuzzy Edge Labeling (PTARFEL) on interval-valued fuzzy graphs was proposed and studied. The algorithm adopts the product-based approach to find the membership value of edge membership values, taking into account the total of membership values and the range of the membership values of the incident vertex. Such an approach makes the labeling process more flexible than the additive fuzzy labeling process. A fuzzy graph can be converted into an interval-valued fuzzy labeling graph through ε -neighborhood intervals. Such conversion will keep the membership values in an interval format while having different conditions of labeling for vertices and edges. The main findings of this study are that all the fuzzy graphs with Product of Total and Range Fuzzy Edge Labeling can be changed into Interval Valued Product of Total and Range Fuzzy Edge Labeling graphs. Next, the proposed labeling system was tested on some significant graphs like star graphs, path graphs, and odd cycle graphs. It is the analytical insight that selects the value for ε that plays an important role in maintaining mutually disjoint interval memberships. If ε is selected less than half the value of the smallest possible difference between two different membership values, the intervals will not overlap, but it is possible to have intervals that do overlap if ε is set larger. This points to the need for interval construction that is controlled in terms of the consistency of labels. In general, the results validate that PTARFEL is a generalized and meaningful model of uncertainty for interval-valued fuzzy graph structure. This method improves the fuzzy graph labeling theory, in which interval uncertainty is added to the product-based edge membership formulation of a fuzzy graph. This labeling technique can be generalized for other graph classes, such as complete graphs, bipartite graphs, wheel graphs, flower graphs, etc., in future work. In the future, algorithmic implementation, comparison with the already developed fuzzy labeling techniques, and applications in communication networks, decision-making systems, optimization models, and uncertain data representation will be studied.

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